INTEGRAZIONE

TEOREMA FONDAMENTALE

$$\int f(x)\,dx = F(x)+c \Rightarrow \int_a^b f(x)\,dx = F(b)-F(a)$$

PROPRIETÀ GENERALI

$a<b<c \Rightarrow \int_a^c = \int_a^b + \int_b^c$	$\left\lvert \int_a^b f \right\rvert \le \int_a^b \lvert f\rvert$
$\int_a^b = -\int_b^a \qquad \int_a^a = 0$	$\left\lvert \int_a^b f\cdot g \right\rvert \le \sup_{[a,b]} \lvert f\rvert \cdot \int_a^b \lvert g\rvert$
$f \le g \Rightarrow \int_a^b f \le \int_a^b g$	$\left\lvert \int_a^b f\cdot g \right\rvert \le \sqrt{\int_a^b f^2}\cdot\sqrt{\int_a^b g^2}$ *(disug. di Cauchy-Schwarz)*
$f\in\mathscr{C} \Rightarrow \int_a^b f = f(\xi)(b-a)$ *(teo valor medio)*	$f\in\mathscr{C}, g\ge 0 \Rightarrow \int_a^b f = f(\xi)\int_a^b g$
f_n integr. $\underset{\text{unif.}}{\to} f$ $\Rightarrow f$ integr., $\int_a^b f_n \to \int_a^b f$	f_n integr., $\Sigma f_n \underset{\text{unif.}}{\to} f$ $\Rightarrow \sum \int_a^b f_n = \int_a^b f$

METODI DI INTEGRAZIONE

- ***per combinazione lineare***
$$\int [\alpha f(x)+\beta\, g(x)]\,dx = \alpha\int f(x)\,dx + \beta\int g(x)\,dx$$
- ***per parti***
$$\int_a^b f'(x)\,g(x)\,dx = f(x)\,g(x)\big|_a^b - \int_a^b f(x)\,g'(x)\,dx$$
- ***per sostituzione***
$$\int_a^b f(\varphi(x))\underbrace{\varphi'(x)dx}_{d\varphi(x)} = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt$$
- ***per decomposizione algebrica***
riconducendo con artifizi a integrali immediati

ESEMPI

- trasformazioni goniometriche
- $\int \frac{P(x)}{Q(x)}dx = \int \frac{Q(x)+p(x)}{Q(x)}dx = \int 1 + \frac{p(x)}{Q(x)}dx$
- $\int \frac{P(x)}{Q(x)}dx = \int \frac{kQ'(x)+\overbrace{p(x)}^{P-kQ'}}{Q(x)}dx = \frac{1}{k}\ln Q(x)+\ldots$
- $\int \frac{dx}{x^2+2bx+c} = \int \frac{dx}{(x+b)^2+(c-b^2)} \Rightarrow x+b=t$

FUNZIONI RAZIONALI FRATTE PROPRIE ($\partial P < n = \partial Q$)

ELEMENTI SEMPLICI

1 $\int \frac{A}{x-a}dx = A\ln\lvert x-a\rvert$

2 $\int \frac{A}{(x-a)^n}dx = A\frac{(x-a)^{1-n}}{1-n} \qquad n\ge 2$

$$F = x^2+px+q$$

3 $\int \frac{ax+b}{F}dx \underset{\Delta<0}{=} \frac{a}{2}\ln F \frac{2b-ap}{\sqrt{4q-p^2}}\operatorname{arctg}\frac{2x+p}{\sqrt{4q-p^2}}$

4 $\int \frac{ax+b}{F^n}dx \underset{\Delta<0}{=} -\frac{a}{2(n-1)F^{n-1}} + \frac{b-a\,p/2}{k^{2n-1}}\underbrace{\int\frac{dt}{(1+t^2)^n}}_{J_n}$

con $k=\sqrt{q-\frac{p^2}{4}}, x+\frac{p}{2}=kt \Rightarrow t=\frac{2x+p}{2k}$ e $J_{n+1}=\frac{t}{2n(1+t^2)^n}+\frac{2n-1}{2n}J_n$

- ***n rad. distinte:*** $Q(x)=(x-\alpha_1)\ldots(x-\alpha_n)$
$$R(x)=\frac{P(x)}{Q(x)}=\frac{C_1}{(x-\alpha_1)}+\ldots+\frac{C_n}{(x-\alpha_n)}$$
- ***r rad. multiple:*** $Q(x)=(x-\alpha_1)^{m_1}\ldots(x-\alpha_r)^{m_r}, \Sigma m_i = n$
$$R(x)=\frac{C_{1,1}}{(x-\alpha_1)}+\ldots+\frac{C_{m_1,1}}{(x-\alpha_1)^{m_1}}+\ldots+\frac{C_{r,1}}{(x-\alpha_r)}+\ldots+\frac{C_{m_r,1}}{(x-\alpha_r)^{m_r}}$$

Troviamo le costanti C_{ij} ad es. per $Q(x)=(x-1)^3(x+2)^2$

PRIMO MODO

$$\frac{P(x)}{\underbrace{(x-1)^3}_{(x-\alpha)^m}\underbrace{(x+2)^2}_{q(x)}} = \frac{C}{\underbrace{(x-1)^3}_{(x-\alpha)^m}} + \frac{p(x)}{\underbrace{(x-1)^2}_{(x-\alpha)^{m-1}}\underbrace{(x+2)^2}_{q(x)}}$$

che equivale a

$$P(x)=C\cdot q(x)+p(x)\cdot(x-1) \Rightarrow p(x)=\text{K}$$

da cui $C=P(1)/q(1)$ ovvero in generale

$$\boxed{C=P(\alpha)/q(\alpha)}$$

si continua ricorsivamente a secondo membro.

SECONDO MODO

$$\frac{P(x)}{Q(x)}=\frac{A_0}{(x-1)^3}+\frac{A_1}{(x-1)^2}+\frac{A_2}{x-1}+\frac{B_0}{(x+2)^2}+\frac{B_1}{x+2}$$

$\times(x-1)^3$ si ha una somma di Taylor per qualche $g(x)$:

$$\frac{P(x)}{(x+2)^2}=A_0+A_1(x-1)+A_2(x-1)^2+g(x)(x-1)^3, \text{ con}$$

$$A_i=\frac{1}{i!}\frac{d^i}{dx^i}\frac{P(x)}{(x+2)^2}\bigg|_{x=1}$$

e analogamente

$$B_i=\frac{1}{i!}\frac{d^i}{dx^i}\frac{P(x)}{(x-1)^3}\bigg|_{x=-2}$$

FUNZIONI BINOMIE

- $\int f(a+bx)dx \quad \Rightarrow a+bx=t$
- $\int x^m(a+bx^n)^p dx$ con $m,n,p\in\mathbb{Q} \mapsto \int R(t)dt$ ponendo:
- $p\in\mathbb{Z} \quad \Rightarrow x=t^k$ con $k=mcm(m,n)$
- $\frac{m+1}{n}\in\mathbb{Z} \quad \Rightarrow a+bx^n=t^d$ con $p=\frac{n}{d}$
- $\frac{m+1}{n}+p\in\mathbb{Z} \quad \Rightarrow \frac{a+bx^n}{x^n}=t^d$ con $p=\frac{n}{d}$

FUNZIONI IRRAZIONALI

$$\int f\left(\sqrt[p]{\left(\frac{ax+b}{cx+d}\right)^\alpha}, \sqrt[q]{\left(\frac{ax+b}{cx+d}\right)^\beta}, \ldots\right)dx$$

$$\frac{ax+b}{cx+d}=t^m \qquad \text{con } m=mcm(p,q,\ldots)$$

casi particolari:

$$\int f\left(\sqrt[p]{(ax+b)^\alpha}, \sqrt[q]{(ax+b)^\beta}, \ldots\right)dx$$

$$\int f\left(\sqrt[p]{x^\alpha}, \sqrt[q]{x^\beta}, \ldots\right)dx$$

- $\int R\left(\sqrt[n]{ax+b}, x\right)dx = \int R\left(t, \frac{t^n-b}{a}\right)\frac{n}{a}t^{n-1}dt$
- $\int R\left(\sqrt{ax^2+2bx+c}, x\right)dx =$
$$= \int R\left(\underset{t-x\sqrt{}}{\pm\sqrt{a}}\frac{at^2+2bt+c}{2(b+at)}, \frac{at^2-c}{2(b+at)}\right)\frac{a(at^2+2bt+c)}{2(b+at)^2}dt$$

ovvero

$$\overset{t=\alpha x+\beta}{=}\begin{cases}\int R\left(\sqrt{t^2+1}\right)\ldots\,dt \overset{t=\operatorname{senh}u}{=} \int R(\cosh u)\ldots\,du\\ \int R\left(\sqrt{t^2-1}\right)\ldots\,dt \overset{t=\cosh u}{=} \int R(\operatorname{senh}u)\ldots\,du\\ \int R\left(\sqrt{1-t^2}\right)\ldots\,dt \overset{t=\operatorname{sen}u}{=} \int R(\cos u)\ldots\,du\end{cases}$$

ovvero

- $\int f\left(\sqrt{ax^2+bx+c}\right)dx$ a $\int R(t)$ ponendo:

$a>0 \quad \sqrt{ax^2+bx+c} = \pm\sqrt{a}\,x+t$

$a<0,\quad \Delta\le 0 \quad \Rightarrow f<0$ sempre

$\Delta>0 \quad \Rightarrow f>0$ per $x_1<x<x_2$: $\dfrac{x-x_1}{x-x_2}=t^2$

$$x=\frac{x_1+x_2t^2}{1+t^2},\ dx=\frac{2(x_2-x_1)t}{(1+t^2)^2}dt$$

- $\int f\left(\dfrac{1}{a+x^2}\right)dx \overset{t=x/\sqrt{a}}{=} f\left(\dfrac{1}{\sqrt{a}}\operatorname{arctg} t\right)$
- $\int f\left(\dfrac{1}{\sqrt{a+x^2}}\right)dx \overset{t=x/\sqrt{a}}{=} f(\operatorname{arcsen} t)$

ESPONENZIALE

- $\int R(e^{\lambda x})dx=\int R(t)\dfrac{1}{\lambda t}dt$

FUNZIONI GONIOMETRICHE

- $\int R(\operatorname{sen}x,\cos^2 x)\cos x\,dx \overset{\operatorname{sen}x=t}{=} \int R(t,1-t^2)dt$
- $\int R(\operatorname{sen}^2 x,\cos x)\operatorname{sen}x\,dx \overset{\cos x=t}{=} -\int R(1-t^2,t)dt$
- $\int R(\operatorname{sen}^2 x,\cos^2 x,\operatorname{sen}x\cos x)dx \overset{\operatorname{tg}x=t}{=}$

$$=\int R\left(\frac{t^2}{1+t^2},\frac{1}{1+t^2},\frac{t}{1+t^2}\right)\frac{1}{1+t^2}dt$$

- $\int R(\operatorname{tg}x)dx \overset{\operatorname{tg}x=t}{=} \int R(t)\dfrac{1}{1+t^2}dt$
- $\int R(\operatorname{sen}x,\cos x)dx \overset{\operatorname{tg}x/2=t}{=} \int R\left(\dfrac{2t}{1+t^2},\dfrac{1-t^2}{1+t^2}\right)\dfrac{2}{1+t^2}dt$

AREA DI UNA REGIONE PIANA

- delimitata da $y=f(x)$, $f\in\mathscr{C}$ e ≥ 0 e l'asse x in $[a,b]$:

$$\boxed{\mathscr{A}=\int_a^b f(x)dx}$$

- delimitata dalle curve $y=f_1(x)<y=f_2(x)$ in $[a,b]$:

$$\mathscr{A}=\int_a^b[f_2(x)-f_1(x)]dx$$

- curva frontiera: in eq. parametriche $x=x(t)$, $y=y(t)$

$$\mathscr{A}=\tfrac{1}{2}\int_\gamma(x\,dy-y\,dx)$$

- curva frontiera: in coord. polari $\rho=\rho(\vartheta)$, $\vartheta_1\le\vartheta\le\vartheta_2$

$$\mathscr{A}=\tfrac{1}{2}\int_{\vartheta_1}^{\vartheta_2}\rho^2(\vartheta)d\vartheta$$

- polarm. normale delimitata da $\rho=\rho_1(\vartheta)$ e $\rho=\rho_2(\vartheta)$

$$\mathscr{A}=\tfrac{1}{2}\int_{\vartheta_1}^{\vartheta_2}[\rho_2^2(\vartheta)-\rho_1^2(\vartheta)]d\vartheta$$

AREA CERCHIO

- in coord. cartesiane, calcolando 4 settori, con $r=1$:

$$\mathscr{A}=4\int_0^1\sqrt{1-x^2}dx=4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\operatorname{arcsen}x\right]_0^1=4\pi$$

- in coord. polari, calcolcando triangolini di area dS:

$$\mathscr{A}=\int_C dS=\int_0^{2\pi}\frac{r^2d\vartheta}{2}=\frac{r^2}{2}\int_0^{2\pi}d\vartheta=\frac{r^2}{2}\vartheta\Big|_0^{2\pi}=r^2\pi$$

LUNGHEZZA DI UNA CURVA γ

$$\boxed{\mathscr{L}=\int_\gamma ds \qquad ds^2=dx^2+dy^2(+dz^2)}$$

curva piana regolare

- di eq. parametriche $x=x(t)$, $y=y(t)$ in $[t_1,t_2]$

$$ds^2=[x'^2(t)+y'^2(t)]dt^2 \quad\Rightarrow\quad \mathscr{L}=\int_{t_1}^{t_2}\sqrt{x'^2(t)+y'^2(t)}\,dt$$

- definita da $y=f(x)$ in $[a,b]$, $dy=f'(x)dx$:

$$ds^2=1+f'^2(x)dx^2 \quad\Rightarrow\quad \mathscr{L}=\int_a^b\sqrt{1+f'^2(x)}\,dx$$

- di coord. polari $x=\rho\cos\vartheta$, $y=\rho\operatorname{sen}\vartheta$

$$ds^2=d\rho^2+\rho^2d\vartheta^2 \quad\Rightarrow\quad \mathscr{L}=\int_\gamma\sqrt{d\rho^2+\rho^2d\vartheta^2}$$

- di eq. polare $\rho=\rho(\vartheta)$

$$ds^2=[\rho^2(\vartheta)+\rho'^2(\vartheta)]d\vartheta \quad\Rightarrow\quad \mathscr{L}=\int_{\vartheta_1}^{\vartheta_2}\sqrt{\rho^2(\vartheta)+\rho'^2(\vartheta)}\,d\vartheta$$

LUNGHEZZA CIRCONFERENZA

- in coord. cartesiane, calcolando 4 archi, con $r=1$:

$$y=\sqrt{1-x^2}\Rightarrow y'^2=x^2/(1-x^2)$$

$$\mathscr{L}=4\underbrace{\int_0^1\sqrt{1+y'^2}dx}_{\operatorname{arcsen}1}=4\underbrace{\left[\operatorname{arctg}\frac{x}{\sqrt{1-x^2}}\right]_0^1}_{\lim\limits_{x\to1^+}(\ )-(\ )_0}=4\left[\frac{\pi}{2}-0\right]=2\pi$$

- in coord. polari, con $\rho=r=cost.$:

$$\mathscr{L}=\int_0^{2\pi}r\,d\vartheta=r\,\vartheta\Big|_0^{2\pi}=2\pi r$$

curva sghemba

- di eq. parametriche $x=x(t)$, $y=y(t)$, $z=z(t)$ in $[t_1,t_2]$

$$ds^2=[x'^2(t)+y'^2(t)+z'^2(t)]dt^2 \quad\Rightarrow\quad \mathscr{L}=\int_{t_1}^{t_2}\sqrt{x'^2(t)+y'^2(t)+z'^2(t)}\,dt$$

- di equaz. $y=y(x)$, $z=z(x)$ in $[a,b]$

$$ds^2=[1+y'^2(x)+z'^2(x)]dx^2 \quad\Rightarrow\quad \mathscr{L}=\int_a^b\sqrt{1+y'^2(x)+z'^2(x)}\,dx$$

- coord. sfer. $x=\rho\cos\varphi\operatorname{sen}\vartheta$, $y=\rho\operatorname{sen}\varphi\operatorname{sen}\vartheta$, $z=\rho\cos\vartheta$

$$ds^2=d\rho^2+\rho^2d\vartheta^2+\rho^2\operatorname{sen}^2\vartheta\,d\varphi^2 \quad\Rightarrow\quad \mathscr{L}=\int_\gamma\sqrt{d\rho^2+\rho^2d\vartheta^2+\rho^2\operatorname{sen}^2\vartheta\,d\varphi^2}$$

VOLUME DI UN SOLIDO

- tra la superf. $z=f(z,y)\ge 0$ e il piano xy entro R:

$$\boxed{\mathscr{V}=\int_R f(x,y)dR}$$

- R normale all'asse x (y):

$$\mathscr{V}=\int_R f(x,y)dR=\int_a^b\int_{f_1(x)}^{f_2(x)}f(x.y)dy\,dx$$

$$\mathscr{V}=\int_R f(x,y)dR=\int_c^d\int_{x_1}^{x_2}f(x.y)dx\,dy$$

- area $\cap$ col piano $x=cost.$:

$$\mathscr{A}=\int_a^b f(x,y)dy$$

- solido normale a xy dato da $z=z_1(x,y)$, $z=z_2(x,y)$:

$$\mathscr{V}=\int_R[z_2(x,y)-z_1(x,y)]dR$$

- solido di rotazione di eq. $r=\varphi(z)$:

$$\mathscr{V} = \pi \int_{z_1}^{z_2} \varphi^2(z)\,dz$$

VOLUME SFERA

- tagliandola in dischi spessi dx e raggio $a=\sqrt{r^2-x^2}$:

$$\mathscr{V}=\int_{sf.} dV=\int_{sf.} a^2\pi\, dx=\int_{-r}^{+r}(r^2-x^2)\pi\, dx=\pi\left[xr^2-\frac{x^3}{3}\right]_{-r}^{+r}=\frac{4\pi r^3}{3}$$

AREA DI UNA SUPERFICIE CURVA

- regolare di eq. $z=f(x,y)$, $f\in\mathscr{C}^1(R)$:

$$\mathscr{A} = \int_R \sqrt{1+f_x^2+f_y^2}\,dR$$

- di rotazione di eq. $r=\varphi(z)$:

$$\mathscr{A} = 2\pi\int_{z_0}^{z_1} r\,ds_m = 2\pi\int_{z_0}^{z_1}\varphi(z)\sqrt{1+\varphi'^2(z)}\,dz$$

- di eq. $x=x(u,v), y=y(u,v), z=z(u,v)$ con le funzioni definite e derivabili in R e con jacobiana di rango 2 in R:

$$\mathscr{A} = \int_R \sqrt{EG-F^2}\,du\,dv$$

con

$$E = x_u^2+y_u^2+z_u^2$$
$$G = x_v^2+y_v^2+z_v^2$$
$$F = x_u x_v + y_u y_v + z_u z_v$$

- di eq. parametr. $x=u\cos v, y=u\,\mathrm{sen}\,v, z=\varphi(u)$:

$$E = 1+\varphi'^2(u)$$
$$G = u^2,\ F=0$$

$$\mathscr{A} = \int_R \sqrt{u^2[1+\varphi'^2(u)]}\,du\,dv = \int_0^{2\pi} dv\int_{u_0}^{u_R}\sqrt{1+\varphi'^2(u)}\,du =$$
$$= 2\pi\int_\gamma u\,du_m$$

con γ curva mediana e du_m suo differenziale d'arco.

BARICENTRO $B(x_B, y_B)$

DEFINIZIONE

- per due masse m_1, m_2 poste in x_1, x_2 :

$$(m_1+m_2)x_B = m_1x_1+m_2x_2$$

- per densità variable con continuiità e massa $m(x)dx$:

$$x_B\int_a^b m(x)\,dx = \int_a^b x\,m(x)\,dx$$

Oss. Per X variabile aleatoria con funzione densità $f(x)$ t.c. $\int_{-\infty}^{+\infty} f(x)\,dx=1$, $x_B=\int_{-\infty}^{+\infty} x\,f(x)\,dx$ è la **media** di X.

- ***curva γ:***

$$x=x(t), y=y(t) \Rightarrow ds^2=[x'^2(t)+y'^2(t)]dt^2$$

$$x_B=\frac{1}{\mathscr{L}}\int_\gamma x\,ds=\frac{1}{\mathscr{L}}\int_{t_1}^{t_2} x(t)\sqrt{x'^2(t)+y'^2(t)}\,dt$$
$$y_B=\frac{1}{\mathscr{L}}\int_\gamma y\,ds=\frac{1}{\mathscr{L}}\int_{t_1}^{t_2} y(t)\sqrt{x'^2(t)+y'^2(t)}\,dt$$

$$y=f(x),\ a\le x\le b \Rightarrow ds^2=[1+f'^2(x)]dx^2$$

$$x_B=\frac{1}{\mathscr{L}}\int_a^b x\sqrt{1+f'^2(x)}\,dx$$
$$y_B=\frac{1}{\mathscr{L}}\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$$

- ***regione R :*** $x_B=\dfrac{1}{\mathscr{A}}\int_R x\,dR \qquad y_B=\dfrac{1}{\mathscr{A}}\int_R y\,dR$

- ***solido T :*** $x_B=\dfrac{1}{\mathscr{V}}\int_T x\,dT \qquad y_B=\dfrac{1}{\mathscr{V}}\int_T y\,dT$ **??**

SOLIDI DI ROTAZIONE

Data $y=f(x)\ge 0$ che ruota intorno all'asse x in $[a,b]$:

$$\mathscr{V}=\pi\int_a^b f^2(x)\,dx,\quad \mathscr{S}=2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx$$

eq. parametriche

$$\mathscr{S}=2\pi\int_{t_1}^{t_2} y(t)\sqrt{x'^2(t)+y'^2(t)}\,dt$$

SUPERFICIE SFERA

- data dalla rotazione di una semicirconferenza:

$$y=\sqrt{r^2-x^2},\quad y'=\frac{-x}{\sqrt{r^2-x^2}},\quad \sqrt{1+y'^2}=\frac{r}{\sqrt{r^2-x^2}}$$

$$\mathscr{S}=2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\,\frac{r}{\sqrt{r^2-x^2}}\,dx=2\pi r\int_{-r}^{r}dx=4\pi r^2$$

TEOREMI DI GULDINO

- *Area **superifice di rotazione** generata da una linea che ruota attorno a un asse esterno = lungh. linea × lungh. circonf. descritta dal baricentro.*
- *Volume **solido di rotazione** generato da una regione piana che ruota attorno a un asse esterno = area regione × lungh. circonf. descritta dal baricentro.*

INTEGRALI NON RISOLUBILI ANALITICAMENTE

- ***spirale di Fresnel***

$$x(t)=\int_0^t \cos u^2 du,\quad y(t)=\int_0^t \mathrm{sen}\,u^2 du$$
$$x(\infty)=y(\infty)=\sqrt{\pi/2}/2$$

- ***ellittici primo tipo***

$$F(\varphi|m)=\int_0^\varphi 1/\sqrt{1-m\,\mathrm{sen}^2\vartheta}\;d\vartheta$$
$$K(m)=F(\pi/2|m): \text{i.e. completo}$$

- ***ellittici secondo tipo***

$$E(\varphi|m)=\int_0^\varphi \sqrt{1-m\,\mathrm{sen}^2\vartheta}\;d\vartheta$$
$$E(m)=E(\pi/2|m): \text{i.e. completo}$$

- ***funzione Z di Jacobi***

$$Z(\varphi|m)=E(\varphi|m)-E(m)F(\varphi|m)/K(m)$$