Prove di CLIL //

What is Determinant in a Matrix?

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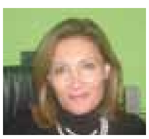


[CATERINA OLIVERIO]
Caterina Oliverio, laureata in Lingue e Letterature Straniere presso l'Università di Bari, è docente di Lingua e Letteratura Inglese nelle scuole superiori dal 1992. Dal 2000 insegna al Liceo scientifico "Luigi Siciliani" di Catanzaro, dove si occupa di English for Specific Purposes. Collabora a contratto con la Facoltà di Medicina e Chirurgia dell'Università "Magna Graecia" di Catanzaro.

Topics needed
- definition of a matrix
- algebraic operations of matrices
- calculation of a determinant

Objectives
- verifying some properties of the determinant
- solving a simple linear system
- Cramer’s rule

[DETERMINANT PROPERTIES]
Let’s verify some properties of the Determinant. In order to do this, we use a 3×3 matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 4 \\ -2 & 0 & 3 \end{pmatrix}$$

whose determinant is $|A| = 13$

Property 1. $|A_T| = |A|$ (the determinant doesn't change by transposing the matrix)
In fact:

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -2 & -1 & 0 \\ 0 & 4 & 3 \end{pmatrix} \quad \text{and} \quad |A_T| = 13$$

Property 2. $|kA| = k^n\,|A|$ (if we multiply the matrix by a not null scalar, its determinant is multiplied by k^n)
In fact:

$$2A = \begin{pmatrix} 2 & -4 & 0 \\ 0 & -2 & 8 \\ -4 & 0 & 6 \end{pmatrix} \quad \text{and} \quad |2A| = 104 = 2^3\,|A|$$

Property 3. $|A \cdot B| = |A| \cdot |B|$ (**Binet's rule**: the determinant of the product is equal to the product of the determinants)
In fact, if we assume

$$B = 2A, \quad \text{with } |B| = 104$$

it follows that

$$A \cdot B = \begin{pmatrix} 2 & 0 & -16 \\ -16 & 2 & 16 \\ -16 & 8 & 18 \end{pmatrix} \quad \text{and} \quad |A \cdot B| = 1352$$

where

$$|A| \cdot |B| = 13 \cdot 104 = 1352$$

For this example, we note that

$$A \cdot B = A \cdot 2A = 2A^2$$

and then

$$|A \cdot B| = |2A^2| = 2^3|A|^2 = 8 \cdot 169 = 1352$$

[A SIMPLE SAMPLE]

Now we would like to investigate why we call it a "determinant". That is, what a determinant is meant to determine. Let's consider the following linear system:

$$\begin{cases} 2x - 3y = 0 \\ -x + 4y = 1 \end{cases}$$

We can easily write it in a compact form by using matrices:

$$\underbrace{\begin{pmatrix} 2 & -3 \\ -1 & 4 \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\underline{b}}$$

In general, we note that:

- the rows of the matrix A correspond to the equations of the system
- the columns of the matrix A are as many as the unknowns of the system itself.

When the number of equations of the system are equal to the number of its unknowns, its matrix A is a square one, and so we can calculate its determinant.

A solution of our linear 2×2 system is every couple $\underline{s} = (\alpha, \beta)_T$ such that $A \cdot \underline{s} = \underline{b}$.

In 1750, the Genevan mathematician **Gabriel Cramer** found an outermost method to solve linear "square" systems $A \cdot \underline{x} = \underline{b}$ with $|A| \neq 0$. In order to determine the first unknown, we need to replace the first column of A with the vector $\underline{b}$; then, we calculate the determinant of the resultant matrix A_1 and divide it by $|A| \neq 0$. Similarly, we go on building A_2 with reference to the second unknown and so on…

$$x = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 0 & -3 \\ 1 & 4 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}} = \frac{3}{5}$$

and

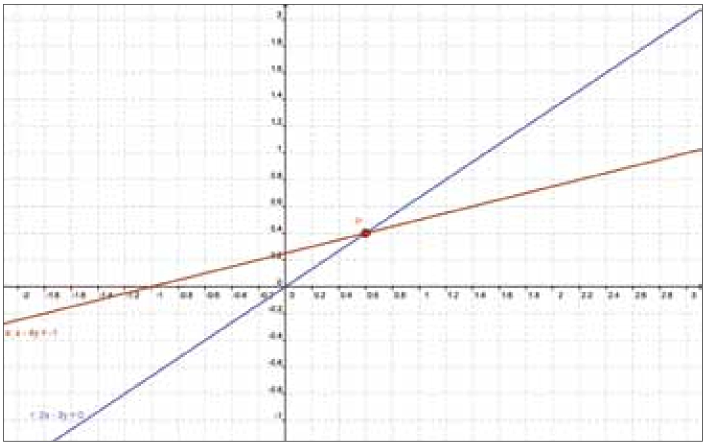
$$y = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & -3 \\ -1 & 4 \end{vmatrix}} = \frac{2}{5}$$

Actually

$$\begin{cases} 2\,\frac{3}{5} - 3\,\frac{2}{5} = \frac{6-6}{5} = 0 \\ -\,\frac{3}{5} + 4\,\frac{2}{5} = \frac{-3+8}{5} = 1 \end{cases}$$

So $\underline{s}_T = (3/5, 2/5)$ comes to be a solution of our system. Graphically, we can easily verify that it is the ONLY solution of our system: the point P = (3/5, 2/5) is the unique intersection between the two straight lines $2x - 3y = 0$ and $-x + 4y = 1$.

A system having a single unique solution is usually called a DETERMINED system. And here comes what a determinant is meant to determine: the unique solution of a square linear system whose matrix A has a nonzero determinant.



[GLOSSARY]

1. algebraic operations of matrices : sum, product with a scalar, product between two matrices

2. 3×3 matrix : a 3-by-3 matrix has three rows and three columns

3. transposing : we get the transpose matrix of A by switching its rows with its columns and vice versa

4. linear system m×n : a set of m linear equations, involving the same n variables, that have to be simultaneously satisfied

5. 2×2 system : a linear "square" system with 2 equations and 2 unknowns; it has a 2×2 matrix (2 rows and 2 columns)

[AND FINALLY...]

Who is determinant in "Matrix"?

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